Topological censorship from the initial data point of view

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Abstract

We introduce a natural generalization of marginally outer trapped surfaces (MOTSs) in an initial data set, called immersed MOTSs, and prove that for 3-dimensional asymptotically flat initial data sets (V, h, K), either V is diffeomorphic to $\mathbb{R}^3$ or V contains an immersed MOTS. We also establish a generalization of the Penrose singularity theorem which shows that the presence of an immersed MOTS generically implies the null geodesic incompleteness of any spacetime that satisfies the null energy condition and which admits a non-compact Cauchy surface. Thus the former result may be seen as a purely initial data version of the Gannon-Lee singularity theorem. It can also be viewed as a non-time-symmetric version of a theorem of Meeks-Simon-Yau which implies that any asymptotically flat Riemannian 3-manifold that is not diffeomorphic to $\mathbb{R}^3$ contains an embedded stable minimal surface. As the Gannon-Lee singularity theorem may be viewed as a precursor to the space-time principle of topological censorship, we go further to obtain an initial data version of topological censorship. We establish, under a natural physical assumption, the topological simplicity of 3-dimensional asymptotically flat initial data sets with MOTS boundaries. We also obtain a generalization of these results to higher dimensions.

1 Introduction

Topological censorship is a basic principle of spacetime physics. It is a set of results, beginning with the topological censorship theorem of Friedman, Schleich, and Witt [20], that establishes the topological simplicity at the fundamental group level of the domain of outer communications (the region outside all black holes and white holes) under a variety of physically natural circumstances, see e.g. [20, 21, 24, 12]. Topological censorship has played a role in black hole uniqueness theorems, and in particular has been used to determine black hole topology in $3+1$ dimensions, see e.g. [10, 24]. An important precursor to the principle of topological censorship, which serves to motivate it, is the Gannon-Lee singularity theorem [26, 33]. We note here that all of these results are *spacetime* results. They involve conditions, such as global hyperbolicity or the existence of a regular past and future null infinity, that are global in time. From the evolutionary point of view, there is the difficult question of determining whether a given initial data set will give rise to a spacetime satisfying these conditions. In order to separate out the principle of topological censorship from these difficult questions of global evolution, it would be useful to obtain a pure initial data version of topological censorship. This is the aim of the present paper.

We begin by presenting an initial data version of the Gannon-Lee singularity theorem for 3-dimensional initial data sets. (Some related work in higher dimensions has been considered in [42].) The result we obtain is of interest in its own right and may be viewed as a non-time-symmetric version of results of Meeks-Simon-Yau [34]. The approach to the proof taken here relies on recent developments in the theory of marginally outer trapped surfaces (MOTSs). For background on MOTSs we refer the reader to the recent survey article [1]. A brief review of topological censorship, including the result of Gannon and Lee, will be given in Section 2. In order to adopt an initial data point of view on these results, the first new issue that needs to be addressed is to decide what exactly constitutes an initial data singularity theorem. This question is taken up in Section 3. Our discussion leads to the notion of *immersed* MOTSs. In Section 4 we present our initial data version of the Gannon-Lee singularity theorem. In Section 5 we formulate and prove an initial data version of topological censorship for 3-dimensional initial data sets. The final section establishes a related initial data result in higher dimensions.

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2 A brief review of topological censorship

A precursor to topological censorship is the Gannon-Lee singularity theorem. From familar examples, we observe that nontrival topological structures (e.g., throats joining different universes) tend to pinch off and form singularities. This intuitive idea has been captured in the following theorem proved independently by Gannon [26] and Lee [33]:

Theorem 2.1 ([26, 33])**.** *Let (M, g) be a globally hyperbolic spacetime which satisfies the null energy condition,* $\mathrm{Ric}(X, X) \geq 0$ *for all null vectors X, and which contains a Cauchy surface V that is regular near infinity. If V is not simply connected, then (M, g) is future null geodesically incomplete.*

The condition "regular near infinity" is a mild asymptotic flatness condition. Thus, under suitable curvature and causality conditions, non-trivial fundamental group entails the formation of singularities, as indicated by the future null geodesic incompleteness.

The notion of topological censorship may then be described as follows: As the Gannon-Lee theorem suggests, non-trivial topology tends to induce gravitational collapse. According to the weak cosmic censorship conjecture, the process of gravitational collapse leads to the formation of an event horizon which shields the singularities from view. As a result, non-trivial topology should become hidden behind the event horizon, and the domain of outer communications should have simple topology.

This notion was formalized by the topological censorship theorem of Friedman, Schleich, and Witt [20]. Their theorem applies to asymptotically flat spacetimes, i.e. spacetimes admitting a regular null infinity (conformal completion) $\mathscr{I} = \mathscr{I}^+ \cup \mathscr{I}^-$, $\mathscr{I}^\pm \approx \mathbb{R} \times S^2$, such that $\mathscr{I}$ admits a simply connected neighborhood U.

Theorem 2.2 ([20])**.** *Let (M, g) be a globally hyperbolic, asymptotically flat spacetime satisfying the null energy condition. Then every causal curve from $\mathscr{I}^-$ to $\mathscr{I}^+$ can be deformed (with endpoints fixed) to a curve lying in a simply connected neighborhood U of $\mathscr{I}$.*

The conclusion in physical terms asserts that observers traveling from $\mathscr{I}^-$ to $\mathscr{I}^+$ are unable to probe any non-trivial topology.

The domain of outer communications is the region $D = I^-(\mathscr{I}^+) \cap I^+(\mathscr{I}^-)$. The topological censorship theorem of Friedman, Schleich and Witt is really a statement about the domain of outer communications, since any causal curve from $\mathscr{I}^-$ to $\mathscr{I}^+$ is contained in D. Strictly speaking, their theorem does not give any direct information about the topology of the domain of outer communications, because it is a statement about *causal* curves, rather than arbitrary curves. However, in [13], Chruściel and Wald used the Friedman-Schleich-Witt result to prove that for stationary (i.e., steady state) black hole spacetimes, the domain of outer communications is simply connected, see also [30]. Subsequent to the work of Chruściel and Wald [13], the second author

was able to show that the simple connectivity of the domain of outer communications holds in general:

Theorem 2.3 ([21]). *Let (M, g) be an asymptotically flat spacetime such that a neighborhood of $\mathscr{I} = \mathscr{I}^+ \cup \mathscr{I}^-$ is simply connected, Suppose that the domain of outer communications $D = I^-(\mathscr{I}^+) \cap I^+(\mathscr{I}^-)$ is globally hyperbolic and satisfies the null energy condition. Then D is simply connected.*

While the proof of Theorem 2.3 makes use of the Friedman-Schleich-Witt result, the conclusion is actually stronger. Thus, in the asymptotically flat setting, topological censorship can be taken as the statement that the domain of outer communications is simply connected. Topological censorship has been extended in various directions, for example to the setting of asymptotically locally anti de-Sitter spacetimes [24], and more recently to Kaluza-Klein spacetimes [12].

3 Initial data singularity theorems

What is an initial data singularity theorem? In view of the Penrose singularity theorem, any result that proves the existence of trapped surfaces in the initial data set from conditions on the geometry and energy and momentum density of the initial data set qualifies.

We recall some basic definitions. Let (M, g) be a 4-dimensional time-oriented Lorentzian manifold, henceforth referred to as a spacetime. Let (V, h, K) be an initial data set for (M, g), that is, a spacelike hypersurface V of M with induced metric h and second fundamental form K.[1] Let u denote the future pointing timelike normal to V in M. Let Σ be a closed two-sided hypersurface in V. Then Σ admits a smooth unit normal field ν in V that is unique up to sign. We will refer to our choice of ν as the outward pointing unit normal. Then $l_+ = u + \nu$ and $l_- = u - \nu$ are, respectively, the future directed outward and inward null normal vector fields along Σ viewed as a co-dimension 2 submanifold of M. Tracing the null second fundamental forms $\chi_\pm$ associated to the null normals $l_\pm$, one obtains the *null expansion scalars* (or *null mean curvatures*) $\theta_\pm = \mathrm{tr}\,\chi_\pm = \mathrm{div}_\Sigma l_\pm$. Note that $\theta_\pm = \mathrm{tr}_\Sigma K \pm H$, where H is the mean curvature scalar of Σ in V with respect to ν. Our sign convention for H here is such that H is the tangential divergence of ν in $\Sigma \subset V$.

In a region of spacetime where the gravitational field is strong, it can happen that $\theta_- < 0$ and $\theta_+ < 0$ along Σ, in which case Σ is called a *trapped surface*. Focusing attention on the outward null normal only, Σ is said to be an *outer trapped surface* if $\theta_+ < 0$, and is said to be a *marginally outer trapped surface* (MOTS) if θ_+ vanishes identically.

The significance of trapped surfaces stems from their prominent role in Penrose's famous singularity theorem:

[1] While we refer to the ambient spacetime, and give geometric definitions relevant to (M, g), one should note that what follows can be understood solely on the level of the initial data set (V, h, K).

Theorem 3.1 (Penrose singularity theorem, cf. [28, Section 8.2])**.** *Let (M, g) be a spacetime. Suppose that*

(i) M admits a non-compact Cauchy surface V,

(ii) M obeys the null energy condition, and

(iii) V contains a trapped surface Σ.

Then at least one of the future directed null normal geodesics emanating from Σ is incomplete.

Beig and Ó Murchadha [6] have given criteria for vacuum initial data sets to contain a trapped surface. In view of Theorem 3.1, we regard their result as an initial data singularity theorem.

In their influential paper [44], Schoen and Yau have given conditions that imply the existence of a MOTS in an initial data set. Roughly, if enough matter is packed into a small enough region, then a MOTS must be present. Should conditions on an initial data set that imply the existence of a MOTS be viewed as an initial data singularity result? In view of the next result, the answer is yes. We give a version of the Penrose singularity theorem that applies to MOTS; related versions that apply to separating outer trapped surfaces have been considered in [27, 45, 2]; see also [42]. The proof is a variation of the original proof of the Penrose singularity theorem.

Theorem 3.2. *Let (M, g) be a spacetime. Suppose that the following conditions are satisfied:*

(i) M admits a non-compact Cauchy surface V.

(ii) M obeys the null energy condition.

(iii) V contains a MOTS Σ.

(iv) The generic condition[2] holds on each future and past inextendible null normal geodesic η to Σ.

Then at least one of the null normal geodesics to Σ is future or past incomplete.

Proof. We may assume that Σ is connected. Recall that a MOTS Σ is, by definition, two-sided. We distinguish two cases: Either Σ separates V, i.e. $V \setminus \Sigma$ has exactly two components, or Σ does not separate V, in which case $V \setminus \Sigma$ is connected.

Assume that Σ separates V. At least one of the two components of $V \setminus \Sigma$, call it U, is unbounded. By taking the time dual of M if necessary, we may assume that Σ

[2]The generic condition is a mild curvature assumption that asks that there be a point p on each null normal geodesic η as in the statement of the theorem and a vector X at p orthogonal to η' such that, $g(R(X, \eta')\eta', X) \neq 0$. Put differently, we ask that there be a non-zero tidal acceleration somewhere along η. See Theorem 2 in Section 8.2 of [28] and Lemma 2.9 of [4].

satisfies $\theta_+ = 0$ with respect to the null normal $l_+ = u + \nu$, where ν is the smooth unit normal to Σ in V that points into $W = \overline{U}$.

Consider the achronal boundary $\partial J^+(V \setminus U)$. Then $\mathcal{H} := \partial J^+(V \setminus U) \setminus (V \setminus W)$ is a C^0 achronal hypersurface with boundary Σ, generated by null geodesics orthogonal to Σ with initial tangents given by ℓ_+. (We refer the reader to [38] for a thorough treatment of achronal boundaries.) Since the boundary of $\mathcal{H}$ is Σ and because W is non-compact, a standard argument using the integral curves of a global timelike vector field to construct a continuous injective map from $\mathcal{H}$ into W shows that $\mathcal{H}$ is non-compact.

One may now argue as in [19, Theorem 5.2] to obtain an affinely parameterized future inextendible null geodesic generator $\eta : [0, a) \to M$, $a \in (0, \infty]$, of $\mathcal{H}$ such that $\eta(0) = p \in \Sigma$ and $\eta'(0) = \ell_+(p)$. Moreover, since $\mathcal{H}$ is achronal, there are no focal cut points to Σ along η. It follows that η is contained in a smooth null hypersurface $N \subset \mathcal{H}$ generated by null geodesics emanating from Σ near p with initial tangents given by ℓ_+; cf. Section 6 in [31].

Suppose that η is future complete, i.e., that $a = \infty$. For each $t \geq 0$, consider the null Weingarten map $b = b(t)$ of N at $\eta(t)$ with respect to $\eta'(t)$; cf. e.g., [22]. Then $b = b(t)$ satisfies the Riccati-type equation

$$b' + b^2 + \mathcal{R} = 0, \tag{3.1}$$

where $'$ denotes covariant differentiation, and where $\mathcal{R}$ encodes the curvature values $g(R(X, \eta')\eta', Y)$ for X, Y orthogonal to η'. Tracing (3.1) we obtain that

$$\frac{d\theta}{dt} = -\mathrm{Ric}\,(\eta', \eta') - \sigma^2 - \frac{1}{2}\theta^2\,, \tag{3.2}$$

where $\theta := \mathrm{tr}\, b$ is the null mean curvature of N along η, and σ is the trace of the square of the trace-free part of b.

Since Σ is a MOTS, we have that $\theta(0) = 0$. Further, since by the null energy condition $\frac{d\theta}{dt} \leq 0$ for all $t \geq 0$, a standard analysis of (3.2) shows that $\theta(t) \to -\infty$ in finite affine parameter time, unless $\theta = \sigma = 0$ along η. In the latter case, b vanishes along η, so that $\mathcal{R}$ also vanishes by (3.1). This violates the generic condition. Thus η is future null geodesically incomplete.

Now suppose that Σ does not separate V.

We first observe that since V is a Cauchy surface for M, M is diffeomorphic to $\mathbb{R} \times V$. Each covering space $\tilde{V}$ of V with covering map $p : \tilde{V} \to V$ gives rise, in an essentially unique way, to a covering spacetime $\tilde{M}$ of M with covering map $P : \tilde{M} \to M$ so that $\tilde{V}$ is a Cauchy surface for $\tilde{M}$ and $P|_{\tilde{V}} = p$.

Since Σ is two-sided but does not separate V, we can make a cut along Σ to obtain a connected manifold V' with two boundary components, each isometric to Σ. Taking $\mathbb{Z}$ copies of V', and gluing these copies end-to-end we obtain an obvious covering $p : \tilde{V} \to V$ of V. The inverse image $p^{-1}(\Sigma)$ consists of $\mathbb{Z}$ copies of Σ, each one separating $\tilde{V}$. Let $\Sigma_0 \subset \tilde{V}$ denote one of these copies. As per the comment above,

we obtain a covering spacetime $\tilde{M}$, with non-compact Cauchy surface $\tilde{V}$. Since the covering map is a local isometry, the curvature assumptions on M lift to $\tilde{M}$. It follows from our earlier arguments that there exists a null normal geodesic η_0 to Σ_0 that is future or past incomplete. The projection of η_0 under the covering map P will then be a past or future incomplete null normal geodesic to Σ in M. $\square$

Remark 3.3. Theorem 3.2 is false without the generic condition. This can be seen for example by considering a Cauchy surface V for the extended Schwarzschild solution. This surface V will meet the event horizon in a MOTS, Σ, whose null normal geodesics are the generators of the horizon, which are known to be complete. Moreover, the generic condition is easily seen to fail along each null normal geodesic in this setting. However, here the outward null normal geodesics form a totally geodesic null hypersurface. This rigidity is always present in the setting of Theorem 3.2 when the generic condition fails and we have completeness of the null normal geodesics: The family of future or past null geodesics always form a totally geodesic null hypersurface in this case. The details of this will appear elsewhere. In particular, this implies that the generic condition will fail along all of the future or past null geodesics of Σ, and therefore Theorem 3.2 may be strengthened to only require that the generic condition hold along one (future pointing) null normal geodesic ray and one (past pointing) null normal geodesic ray, rather than all of them.

Our view of what should be considered an initial data singularity result must be taken one step further to accommodate other fundamental examples. There is a more general type of object in an initial data set that implies a Penrose-type singularity theorem, which we refer to as an *immersed* MOTS.

Definition 3.4. *Given an initial data set (V, h, K), we say that a subset $\Sigma \subset V$ is an* immersed MOTS *if there exists a finite cover $\tilde{V}$ of V with covering map $p : \tilde{V} \to V$ and a closed embedded MOTS $\tilde{\Sigma}$ in $(\tilde{V}, p^*h, p^*K)$ such that $p(\tilde{\Sigma}) = \Sigma$.*

We say that Σ is a spherical (or toroidal, etc.) immersed MOTS if $\tilde{\Sigma}$ is spherical (or toroidal, etc.).

The best known example of an immersed MOTS (that is not a MOTS) occurs in the so-called $\mathbb{RP}^3$ geon; see e.g. [20] for a detailed description. The $\mathbb{RP}^3$ geon is a globally hyperbolic spacetime that is double covered by the extended Schwarzschild spacetime. Its Cauchy surfaces have the topology of a once punctured $\mathbb{RP}^3$. The Cauchy surface that is covered by the $t = 0$ slice in the extended Schwarzschild spacetime contains a projective plane Σ that is covered by the unique minimal sphere in the Schwarzschild slice. Hence Σ is a spherical immersed MOTS.

Corollary 3.5. *Theorem 3.2 remains valid with "MOTS" replaced by "immersed MOTS".*

Proof. Construct the covering spacetime $\tilde{M}$ corresponding to the covering Cauchy surface $\tilde{V}$ arising in the definition of the immersed MOTS. This spacetime satisfies

the hypothesis of Theorem 3.2 relative to the MOTS $\tilde{\Sigma} \subset \tilde{V}$. Theorem 3.2 then asserts the existence of an incomplete null normal geodesic to $\tilde{\Sigma}$. We then project down to M to obtain the desired result. $\square$

We conclude from Corollary 3.5 that any result that implies the existence of an immersed MOTS in an initial data set should be viewed as an initial data singularity theorem.

4 An initial data version of the Gannon-Lee singularity theorem

Let (V, h, K) be a 3-dimensional asymptotically flat initial data set. Recall that this means that the complement of some compact set in V has finitely many components, each of which is diffeomorphic to $\mathbb{R}^3 \setminus B_1(0)$, and such that in Euclidean coordinates, one has that $h_{ij} - \delta_{ij}$ and K_{ij} decay suitably. We refer the reader to the careful discussion in [11, 15, 18] for a precise definition and references.

Theorem 4.1. *Let (V, h, K) be a 3-dimensional asymptotically flat initial data set. If V is not diffeomorphic to $\mathbb{R}^3$ then V contains an immersed MOTS.*

Theorems 3.2 and 4.1 together imply that if V is an asymptotically flat Cauchy surface in a spacetime that satisfies appropriate curvature conditions and if V has non-trivial topology, then the spacetime is singular. Thus, from the point of view put forth in Section 3, we regard Theorem 4.1 as an initial data version of the Gannon-Lee singularity theorem.

Beig and Ó Murchdha [5], Miao [35], and Yan [47] have constructed examples of asymptotically flat scalar flat time-symmetric initial data on $\mathbb{R}^3$ that contain stable minimal spheres (i.e. a spherical MOTS). Hence initial data may well contain an immersed MOTS *and* be diffeomorphic to $\mathbb{R}^3$.

The reader should compare Theorem 4.1 with the work of Meeks-Simon-Yau [34] (see in particular Theorem $1'$ on page 645 and Proposition 1 on page 650 of that paper), which implies that an asymptotically flat 3-manifold that is not diffeomorphic to $\mathbb{R}^3$ contains an *embedded* stable minimal sphere or projective plane.

Remark 4.2. Recall that the dominant energy condition is said to hold on (V, h, K) provided that $\mu \geq |J|$ along V, where $\mu = \frac{1}{2}(R - |K|^2 + (\text{trace } K)^2)$ and $J = \text{div}(K - \text{trace } K)$. The conclusion of Theorem 4.1 can be refined if one assumes in addition that the dominant energy condition holds. One may then conclude that if V is not diffeomorphic to $\mathbb{R}^3$, then V contains a *spherical* immersed MOTS; cf. Remarks 4.4 and 4.6.

The following existence result for closed MOTSs was proposed by R. Schoen and is based on forcing a blow up of Jang's equation and analysis of the blow up set

as in [43]. It was first proved by Andersson and Metzger [3] in the 3-dimensional case and then by the first author [16, 17] in general dimension (with small singular set in dimension $n \geq 8$) using a different method. The survey article [1] contains an extensive overview of the techniques developed in [43, 3, 16, 17], including a discussion of the geometric properties of the MOTS whose existence is established.

Theorem 4.3 (Cf. Theorem 3.3 in [1])**.** *Let (V, h, K) be an n-dimensional initial data set with $3 \leq n \leq 7$, and let W be a connected compact n-dimensional manifold with boundary in V. Suppose that the boundary ∂W can be expressed as a disjoint union, $\partial W = \Sigma_{in} \cup \Sigma_{out}$, such that $\theta^+ < 0$ along Σ_{in} with respect to the null normal whose projection points into W, and $\theta^+ > 0$ along Σ_{out} with respect to the null normal whose projection points out of W. Then there exists a smooth compact embedded MOTS Σ in W that separates Σ_{in} from Σ_{out}. Moreover, Σ is almost minimizing.*

Remark 4.4. Assume that (V, h, K) in Theorem 4.3 also satisfies the dominant energy condition $\mu \geq |J|$. We claim that there exists a MOTS $\Sigma \subset W$ that admits a metric of positive scalar curvature (not necessarily in the conformal class of the induced metric). Note first that by Theorem 2.1 in [25], the induced metric on any closed stable MOTS in V is conformal to a metric of non-negative constant scalar curvature; if the dominant energy condition $\mu \geq |J|$ is strict at any point on Σ, then the induced metric is conformal to a metric of constant positive scalar curvature. Apply Theorem 18 in [18] to find a sequence of initial data (h_i, K_i) on V converging to the original data (h, K) as $i \to \infty$ and such that (V, h_i, K_i) satisfies the strict dominant energy condition $\mu_i > |J_i|$. It follows that the MOTS $\Sigma_i \subset W$ in (V, h_i, K_i) whose existence is guaranteed by Theorem 4.3 admits a metric of positive scalar curvature. The uniform almost minimizing property of Σ_i shows that a subsequence of Σ_i converges smoothly to a closed embedded stable MOTS $\Sigma \subset W$ in (V, h, K) as $i \to \infty$. This MOTS Σ then has all the asserted properties.

Let (V, h, K) and W be as in the statement of Theorem 4.3. We say that the boundary ∂W is *null mean convex* provided it has positive outward null expansion, $\theta^+ > 0$, and negative inward null expansion, $\theta^- < 0$. We note that round spheres in Euclidean slices of Minkowski space, and, more generally, large "radial" spheres in asymptotically flat initial data sets are null mean convex.

Theorem 4.3 will be used in the following manner:

Proposition 4.5. *Let (V, h, K) be an n-dimensional initial data set with $3 \leq n \leq 7$, and let W be a connected compact n-dimensional manifold with null mean convex boundary ∂W in V. If there are no MOTSs in W, then ∂W is connected.*

Proof. Suppose ∂W is not connected. Then designate one component as Σ_{in} and the union of the others as Σ_{out}. Note that, by the null mean convexity of ∂W, Σ_{in} and Σ_{out} obey the null expansion conditions of Theorem 4.3. Thus there exists a MOTS Σ in W. ∎

Remark 4.6. If the initial data set (V, h, K) in Proposition 4.5 satisfies the dominant energy condition, then we can conclude that ∂W is connected if we assume that there are no MOTSs in W that admit a metric of positive scalar curvature; cf. Remark 4.4.

Proof of Theorem 4.1. Assume that there are no immersed MOTSs in V. We show that V is then diffeomorphic to $\mathbb{R}^3$.

By suitably truncating the ends of V we obtain a compact 3-manifold $W \subset V$ with null mean convex boundary ∂W, the components of which correspond to the ends of V. If V had more than one end, then W would contain a closed embedded MOTS by Proposition 4.5. This contradicts our assumption that V contains no immersed MOTSs. Hence V has exactly one end.

If V were not orientable, we could pass to the orientable double cover $p : \tilde{V} \to V$. Note that $(\tilde{V}, p^*h, p^*K)$ is an asymptotically flat initial data set with two ends. Our assumption that V contains no immersed MOTSs entails that $(\tilde{V}, p^*h, p^*K)$ cannot contain a closed embedded MOTS. By the same line of reasoning as above, this contradicts the conclusion of Proposition 4.5. Hence V is orientable.

It follows that we can express V as a connected sum $V = \mathbb{R}^3 \# N$, where N is a compact orientable 3-manifold.

Assume that N is not simply connected. The work of Hempel [29] in conjunction with the positive resolution of the geometrization conjecture[3] implies that $\pi_1(N)$ is a residually finite group, and as such admits a proper normal subgroup of finite index.[4] In particular, N, and consequently V, admit non-trivial finite covers. Let $p : \tilde{V} \to V$ be a k-sheeted covering of V with $1 < k < \infty$. Then $(\tilde{V}, p^*h, p^*K)$ is an asymptotically flat initial data set with k ends. Our assumption that V does not contain any immersed MOTSs shows that $(\tilde{V}, p^*h, p^*K)$ cannot contain any closed embedded MOTSs. Since $k > 1$, this leads to a contradiction with the conclusion of Proposition 4.5 as above. Hence N is simply connected.

By the positive resolution of the Poincaré conjecture,[3] N is diffeomorphic to $\mathbb{S}^3$. Hence V is diffeomorphic to $\mathbb{R}^3$, as asserted. ☐

Remark 4.7. Contrary to the example of the $\mathbb{RP}^3$ geon discussed earlier, we do not in general expect an immersed MOTS to cover an embedded submanifold in V. To illustrate this point, we consider the following $2+1$ dimensional toy model of the $\mathbb{RP}^3$ geon:

Consider the manifold $\tilde{M} = \mathbb{R}^2 \times S^1$ with coordinates (t, y, θ), $t, y \in \mathbb{R}$, $\theta \in [-\pi, \pi]$. We equip $\tilde{M}$ with the flat Lorentzian metric $\tilde{g} = -dt^2 + dy^2 + d\theta^2$. The slice $t = 0$ is a Cauchy surface for $(\tilde{M}, \tilde{g})$. Geometrically, it is a flat cylinder. Let M be the manifold obtained from $\tilde{M}$ by identifying points via the involution $\psi : (y, \theta) \to (-y, \theta + \pi)$. M may be described as the subset $\{y \geq 0\}$ of $\tilde{M}$, with the points $(t, 0, \theta)$ and $(t, 0, \theta + \pi)$ in the timelike surface $y = 0$ identified. Since $\tilde{g}$ is invariant under ψ, it descends to a flat Lorentzian metric g on M. Hence $(\tilde{M}, \tilde{g})$ is a spacetime double covering of

[3] See [39, 41, 40, 7, 9, 14, 32, 36, 37].

[4] We are grateful to Ian Agol for valuable correspondence on this point.

(M, g). The slice $t = 0$ in M is a Cauchy surface for (M, g). It has the topology of a punctured projective plane.

We now perturb the slice $t = 0$ in $\tilde{M}$. For $a > 0$ sufficiently small, the surface $\tilde{S} = \{(a\sin(2\theta), y, \theta) : y \in \mathbb{R} \text{ and } \theta \in [-\pi, \pi]\} \subset \tilde{M}$ is a Cauchy surface for $(\tilde{M}, \tilde{g})$. The intersection $\tilde{\Sigma}$ of $\tilde{S}$ with the null hypersurface $t = y$ is a circular MOTS, which may be viewed as a perturbation of the MOTS at the intersection of $t = 0$ and $t = y$. Since $\tilde{S}$ is clearly invariant under the involution ψ, it descends under the covering map $p : \tilde{M} \to M$ to a Cauchy surface S for (M, g). Then $\Sigma = p(\tilde{\Sigma})$ is an immersed MOTS in S with transverse self-intersections at the points $(0, 0, 0)$ and $(0, 0, \pi/4)$.

5 An initial data version of topological censorship

As in Theorem 2.2, consider the domain of outer communications $D = I^-(\mathscr{I}^+) \cap I^+(\mathscr{I}^-)$ in a regular black hole spacetime M satisfying the null energy condition. Suppose D is globally hyperbolic, and consider a Cauchy surface for D whose closure V meets the event horizon in a compact surface. As shown in [28, 46], there can be no trapped surface in $V \setminus \partial V$, as otherwise it would be visible at $\mathscr{I}^+$, which is not possible. This remains true even for marginally trapped surfaces, i.e. surfaces for which θ_+ and θ_- are nonpositive; cf., [46, Proposition 12.2.2] and [12, Theorem 6.1]. An argument similar to that of the proof of Theorem 3.2 shows that there can be no immersed MOTS in $V \setminus \partial V$ either; cf. [46, Proposition 12.2.4] and [12, Remark 6.5].

In the context of proving an initial data version of topological censorship, one should think of the initial data manifold V as representing an asymptotically flat slice in the domain of outer communications, whose compact manifold boundary ∂V corresponds to a cross section of the event horizon. On the level of initial data, we represent this cross section by a MOTS ∂V. Our result here is that V has simple topology if each component of the boundary ∂V is spherical and if there are no immersed MOTS in $V \setminus \partial V$:

Theorem 5.1. *Let (V, h, K) be a 3-dimensional asymptotically flat initial data set such that V is a complete manifold whose boundary ∂V is a compact MOTS with respect to the unit normal pointing into V. If all components of ∂V are spherical and if there are no immersed MOTS in $V \setminus \partial V$, then V is diffeomorphic to $\mathbb{R}^3$ minus a finite number of open balls.*

Remark 5.2. The assumption that there are no immersed MOTSs in $V \setminus \partial V$ implies that ∂V is the outermost MOTS of the initial data set. If (V, h, K) is contained in a spacetime such that the dominant energy condition holds in a spacetime neighborhood of ∂V, then the results of [23] show that the components of ∂V must be spherical.

Proof. The proof is similar to the proof of Theorem 4.1. By suitably truncating the asymptotically flat ends of V we obtain a compact 3-manifold $W \subset V$ with boundary

$\partial W = S \cup S'$, where $S = \partial V$ is a MOTS and S' is null mean convex whose components are spherical and correspond to the asymptotically flat ends of V.

Assume that V has more than one end. Let $\Sigma_{in} = S \cup S'_1$ and $\Sigma_{out} = S'_2$, where S'_1 is one of the components of S' and S'_2 is the union of the other components of S'. We have that $\theta_+ > 0$ along Σ_{out} with respect to the outward null normal, as required by Theorem 4.3. As for Σ_{in}, we have that $\theta_+ < 0$ with respect to the inward null normal along S'_1, and that $\theta_+ = 0$ along S. Theorem 4.3 still applies (cf. Section 5 in [3] or Remark 4.1 in [17]) in this situation and shows that there exists a MOTS $\Sigma \subset W$ that is homologous to Σ_{out}. Using also the maximum principle, we see that at least one component of this MOTS Σ must be disjoint from the components of S and hence be contained in $V \setminus \partial V$. This contradicts our assumptions. Thus V has only one end.

The same covering argument as in the proof of Theorem 4.1 shows that V must also be orientable.

It follows that V can be expressed as a connected sum $V = \mathbb{R}^3 \# N$, where N is a compact orientable 3-manifold with boundary $\partial N = S$. Let $\hat{N}$ be the smooth closed orientable 3-manifold obtained by gluing in balls along each of the spherical components of S.

Assume that $\hat{N}$ is not simply connected. Then, exactly as the proof of Theorem 4.1, $\hat{N}$ admits a k-sheeted covering for some $1 < k < \infty$. Since balls and their complements in $\mathbb{R}^3$ are simply connected, it follows that V admits a k-sheeted covering $p : \tilde{V} \to V$. Note that the boundary of $\tilde{V}$ covers the boundary of V, and that it is a MOTS with respect to the pull-back data (p^*h, p^*K). Hence $(\tilde{V}, p^*h, p^*K)$ is an asymptotically flat initial data set with MOTS boundary and k ends. The same argument as above shows that $\tilde{V}$ contains a closed embedded MOTS that is disjoint from its boundary $\partial\tilde{V}$. It follows that $V \setminus \partial V$ contains an immersed MOTS, contrary to our assumption. Thus $\hat{N}$ is simply connected and hence, by the positive resolution of the Poincaré conjecture, diffeomorphic to the sphere.

It follows that $V = \mathbb{R}^3 \# N$ is diffeomorphic to $\mathbb{R}^3$ minus a finite number of open balls, as asserted. $\square$

6 Higher dimensions

The proofs of Theorems 4.1 and 5.1 depend explicitly on certain facts that are specific to three dimensions, specifically the positive resolution of the geometrization conjecture. In this section we present a simple topological condition that implies the existence of immersed MOTSs in asymptotically flat initial data sets of dimension up to seven.

Theorem 6.1. *Let (V, h, K) be an n-dimensional asymptotically flat initial data set, where $3 \le n \le 7$. If V contains an embedded closed non-separating hypersurface*

(which is always the case when the first Betti number $b_1(V)$ of V is positive), then V contains an immersed MOTS.

Proof. As in the proof of Theorem 4.1, we may assume that V has exactly one end and that V is orientable.

Let Σ be an embedded closed non-separating hypersurface in V. We can double V along Σ to obtain a two sheeted covering $p : \tilde{V} \to V$ of V. Note that $(\tilde{V}, p^*h, p^*K)$ is oriented and has two asymptotically flat ends. The same argument as in the proof of Theorem 4.1 shows that this new initial data set contains a closed embedded MOTS, which projects to an immersed MOTS in V.

To see that the assumption $b_1(V) > 0$ implies the existence of a connected closed embedded non-separating hypersurface Σ in V, we supply the following standard topological argument:

Let $\hat{V}$ be the smooth compact orientable manifold obtained from a one point compactification of V. By Mayer-Vietoris, $H_1(\hat{V}, \mathbb{Z})$ is isomorphic to $H_1(V, \mathbb{Z})$, and hence $b_1(\hat{V}) > 0$. Poincaré duality and the fact that there is no torsion in co-dimension one homology implies that $b_1(\hat{V}) > 0$ if and only if $H_{n-1}(\hat{V}, \mathbb{Z}) \neq 0$. By Thom's realizability theorem, see Theorem 11.16 in [8], we may choose a non-vanishing class in $H_{n-1}(\hat{V}, \mathbb{Z})$ that can be represented by a compact embedded hypersurface. Let $\hat{\Sigma}$ be a component of this hypersurface that does not separate $\hat{V}$. (There must be at least one such component for the hypersurface to represent a non-vanishing class in $H_{n-1}(\hat{V}, \mathbb{Z})$.) We may also assume that this non-separating hypersurface $\hat{\Sigma}$ does not pass through the point at infinity of $\hat{V}$. We may hence view $\hat{\Sigma}$ as a smooth closed embedded hypersurface Σ in V. Clearly, Σ does not separate. ☐

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